ANALYTIC PARTIAL WAVE EXPANSION AND INTEGRAL REPRESENTATION OF BESSEL BEAM

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ABSTRACT. This paper describes the partial wave expansion and integral representation of Bessel beams in free space and in the presence of dispersion. The expansion of the Bessel beam wavepacket with constant spectrum is obtained as well. Furthermore, the sum of a triple Legendre polynomial product of same order but different argument follows naturally from the partial wave expansion. The integration of all Bessel beams over all conical angles is shown to have a simple series representation, which confirms the equivalence between the results for both expansion and integral representation.

1. Introduction

The Bessel beam is a mathematical construct that is a solution to wave equation. Bessel beams have attracted attention ever since Durnin [1] published his paper in 1987. The attention is due to the beam's intensity profile that has a sharp peak along the axis of propagation. The electromagnetic Bessel beam also travels with a phase velocity that is higher than speed of light. Numerous authors have used a superposition of Bessel beams in order to form a wavepacket (for example see [2, 3, 4]). Their choice of using an exponential spectrum has allowed them to create a wavepacket that has been called an X-wave. The X-wave however is not a physically possible solution because it carries infinite energy. There is also interest in the superluminal property of X-waves. Two papers have been recently published that clearly show how two schools of thought concerning the X-waves have opposing views. One side states that X-waves are physically realizable, even superluminally [5], whereas the other side states that superluminal effects of X-waves are simply phase related and that superluminal information transfer via X-waves is not possible [6]. The author of this paper shares the latter view. Nevertheless, the mathematics behind Bessel beams, which are building blocks for wavepackets, is very rich as shown in this paper. The mathematical tools developed here may help create a proper wavepacket that has finite energy and is therefore physically realizable. Such a wavepacket may still be of interest because Bessel beam itself has reconstructive properties not recognized in other types of beams [7]. In this paper I will show and confirm two representations of Bessel beam, namely a partial wave expansion and an integral representation. The partial wave expansion of Bessel beam has been reported before, as will be explained later, but the proof presented

here is new and based on a physical argument. The integral representation however is to my knowledge a new result. Furthermore I will show that a simple superposition of Bessel beams with a constant spectrum is suprisingly related to the infinite series of triple Legendre polynomial product. In the process I will show a Bessel beam identity that is simple and also new. I will also briefly describe the above two representations when dispersion is included into wave equation.

2. Mathematical Introduction

A zeroth order Bessel beam that propagates in the z direction can be described by

$$\Phi(\rho, z, t) = \exp(ik_z z - \imath\omega t) J_0(k_\rho \rho) \tag{2.1}$$

where ρ is the transversal distance from the propagation axis z.

Solution (2.1) satisfies the wave equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial\Phi}{\partial t} \tag{2.2}$$

where

$$k_z = \frac{\omega}{c}\cos(\theta) \tag{2.3}$$

$$k_\rho = \frac{\omega}{c}\sin(\theta), \tag{2.4}$$

which is a wave vector construct used by Salo et al. [3].

The angle θ is the axicon angle of the wave. The wave vector $k = (k_z, k_\rho, k_\phi)$ is restricted to lie on a cone described by vertex angle θ. The Bessel beam itself constitutes the sum of all wavevectors on this cone (i.e. sum over all k_ϕ). For simplicity I shall let $c = 1$.

I will first show that the partial wave expansion of a Bessel beam is

$$\begin{aligned}&\exp[\imath\omega\cos(\theta)z - \imath\omega t]J_0[\omega\sin(\theta)\rho] \\ &= \sum_{n=0}^{\infty} 2\imath^n(n+\frac{1}{2})P_n[\cos(\theta)]P_n[\frac{z}{\sqrt{z^2+\rho^2}}]j_n(\omega\sqrt{z^2+\rho^2})\exp[-\imath\omega t],\end{aligned} \tag{2.5}$$

where P_n is Legendre polynomial of order n and j_n is a spherical Bessel function of same order.

I will then show that the integral representation of Bessel beam is

$$\begin{aligned}&\exp[\imath\omega\cos(\theta)z - \imath\omega t]J_0[\omega\sin(\theta)\rho] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} j_0(R)\exp[\frac{\imath z\lambda}{\sqrt{z^2+\rho^2}}]\exp(-\imath\omega t)d\lambda,\end{aligned} \tag{2.6}$$

where j_0 is spherical Bessel function of order zero. R is given by

$$R = \sqrt{\lambda^2 + \omega^2(z^2+\rho^2) - 2\lambda\omega\sqrt{z^2+\rho^2}\cos(\theta)}. \tag{2.7}$$

The Bessel beam wavepacket is a superposition of Bessel beams:

$$\Psi(\rho, z, t) = \int_{-\infty}^{\infty} S(\omega) \exp[\imath\omega \cos(\theta) z - \imath\omega t] J_0[\omega \sin(\theta)\rho] d\omega. \tag{2.8}$$

If $S(\omega) = 1$, I will show that

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp[\imath\omega \cos(\theta) z - \imath\omega t] J_0[\omega \sin(\theta)\rho] d\omega \\ &= \frac{1}{\sqrt{z^2 + \rho^2}} \sum_{n=0}^{\infty} 2\pi(n + \frac{1}{2}) P_n[\cos(\theta)] P_n[\cos(\eta)] P_n[\cos(\gamma)]. \end{aligned} \tag{2.9}$$

where $\cos(\eta) = \frac{z}{\sqrt{z^2+\rho^2}}$ and $\cos(\gamma) = \frac{t}{\sqrt{z^2+\rho^2}}$.

The integral in (2.9) can be evaluated analytically, and from this I will show the result for the following triple Legendre polynomial sum

$$\begin{aligned} &\sum_{n=0}^{\infty} (2n + 1) P_n[\cos(\theta)] P_n[\cos(\eta)] P_n[\cos(\gamma)] \\ &= \frac{2}{\sqrt{\sin^2(\eta)\sin^2(\theta) - [\cos(\eta)\cos(\theta) - \cos(\gamma)]^2}} \end{aligned} \tag{2.10}$$

if $\sin(\eta)\sin(\theta) >\mid \cos(\eta)\cos(\theta) - \cos(\gamma) \mid$, otherwise the sum equals zero.

The result in (2.10) is elegant, short and simply obtained. Similar result has been found by Baranov [19] by using different, more complicated and lenghty method. My result coincides with that of Baranov [19] and I have numerically verified it as well.

I will finally evaluate the following integral:

$$\int_{-1}^{1} J_0[\omega r \sin^2(\theta)] \exp[\imath\omega r \cos^2(\theta)] d[\cos(\theta)] = \sum_{n=0}^{\infty} 2\imath^n j_n(\omega r). \tag{2.11}$$

For this I will use both the partial wave expansion and the integral representation of Bessel beams. In both cases I get the same result and this verifies the equivalence between (2.5) and (2.6). The result itself constitutes a Bessel beam identity which to my knowledge is new.

Marston [8] gives and numerically verifies the Bessel beam partial wave expansion in (2.5). Mitri [9] shows the Bessel beam partial wave expansion directly from the Gegenbauer result published in 1899. I can however find the exact form of the expansion as in (2.5) in the work of Stratton [10, page 413]. The proof of (2.5) in this paper is new, and is described from a physical perspective. The integral representation in (2.6) is to my knowledge new.

3. Bessel Beam Partial Wave Expansion

Let me write a superposition of Bessel beams with varying conical angle θ:

$$\int_{-1}^{1} B[\cos(\theta)]\exp[\imath\omega\cos(\theta)z-\imath\omega t]J_0[\omega\sin(\theta)\rho]d[\cos(\theta)], \tag{3.1}$$

where $B[cos(\theta)]$ is angular spectrum. The angular spectrum can then be expanded in terms of Legendre polynomials, namely

$$B[\cos(\theta)]=\sum_{n=0}^{\infty}a_nP_n[\cos(\theta)], \tag{3.2}$$

where P_n is Legendre polynomial of order n and coefficients a_n are constant.

If I choose the spectrum to be a delta function that eliminates all Bessel beams from the superposition except the one with conical angle θ_0:

$$B[\cos(\theta)]=\delta[\cos(\theta)-\cos(\theta_0)], \tag{3.3}$$

then one has to let $a_n=(n+\frac{1}{2})P_n[\cos(\theta_0)]$, because the standard identity for Legendre polynomials is:

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n[\cos(\theta_0)]P_n[\cos(\theta)]=\delta[\cos(\theta)-\cos(\theta_0)]. \tag{3.4}$$

I let $\alpha=\cos(\theta)$ and $\alpha_0=\cos(\theta_0)$. At $t=0$ and with delta function angular spectrum, (3.1) becomes:

$$\int_{-1}^{1}\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(\alpha_0)P_n(\alpha)\exp(\imath\omega\alpha z)J_0(\omega\sqrt{1-\alpha^2}\rho)d\alpha. \tag{3.5}$$

The expression in (3.5) is equal to the expression for Bessel beam at $t=0$:

$$\begin{aligned}&\exp(\imath\omega\alpha_0 z)J_0(\omega\sqrt{1-\alpha_0^2}\rho)\\ &=\int_{-1}^{1}\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(\alpha_0)P_n(\alpha)\exp(\imath\omega\alpha z)J_0(\omega\sqrt{1-\alpha^2}\rho)d\alpha.\end{aligned} \tag{3.6}$$

The right hand side of (3.6) is very usefull because the integral in (3.6) can be evaluated, (see [10, 11, 12, 13, 14]). I should mention that Neves et al. [11] state that they do not know of any other report of this evaluation result. However Cregg and Svendlindh [12], Koumandos [13] and Dodonov [14] clearly show that the result follows from publications of Gegenbauer at the end of 19th century. Nevertheless, Cregg and Svendlindh [12] state that the result is not generally well known. Stratton [10, page 411] evaluates the

integral in (3.7) in the exact form as well as authors in [11, 12, 13, 14]:

$$\begin{aligned}&\int_{-1}^{1} P_n(\alpha)\exp(\imath\omega\alpha z)J_0(\omega\sqrt{1-\alpha^2}\rho)d\alpha \\ &= 2\imath^n P_n[\frac{z}{\sqrt{z^2+\rho^2}}]j_n(\omega\sqrt{z^2+\rho^2}).\end{aligned} \tag{3.7}$$

Equation (3.6) therefore becomes the partial wave expansion of Bessel beam written below for any time t:

$$\begin{aligned}&\exp(\imath\omega\alpha_0 z-\imath\omega t)J_0(\omega\sqrt{1-\alpha_0^2}\rho) \\ &= \sum_{n=0}^{\infty} 2\imath^n(n+\frac{1}{2})P_n(\alpha_0)P_n[\tfrac{z}{\sqrt{z^2+\rho^2}}]j_n(\omega\sqrt{z^2+\rho^2})\exp(-\imath\omega t).\end{aligned} \tag{3.8}$$

4. Bessel Beam Integral Representation

I can simplify the Bessel beam partial wave expansion in (2.5) at $t=0$ if I let $\beta=\frac{z}{\sqrt{z^2+\rho^2}}$ and $\mu=\omega\sqrt{z^2+\rho^2}$, to obtain

$$\exp(\imath\omega\alpha_0 z)J_0(\omega\sqrt{1-\alpha_0^2}\rho) = \sum_{n=0}^{\infty} 2\imath^n(n+\frac{1}{2})P_n(\alpha_0)P_n(\beta)j_n(\mu). \tag{4.1}$$

The Fourier transform relationship between Legendre polynomial $P_n(\beta)$ and the spherical Bessel function $j_n(\lambda)$ is given by Mehrem [15]:

$$P_n(\beta) = \frac{(-\imath)^n}{\pi}\int_{-\infty}^{\infty} j_n(\lambda)\exp(\imath\beta\lambda)d\lambda. \tag{4.2}$$

I can therefore further rewrite equation (4.1) as

$$\begin{aligned}&\exp(\imath\omega\alpha_0 z)J_0(\omega\sqrt{1-\alpha_0^2}\rho) \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(\alpha_0)j_n(\lambda)j_n(\mu)\exp(\imath\beta\lambda)d\lambda.\end{aligned} \tag{4.3}$$

The spherical Bessel function is defined as $j_n(\lambda) = \sqrt{\frac{\pi}{2\lambda}}J_{n+\frac{1}{2}}(\lambda)$, where $J_{n+\frac{1}{2}}(\lambda)$ is the Bessel function of half integer order. With this, (4.3) becomes

$$\begin{aligned}&\exp(\imath\omega\alpha_0 z)J_0(\omega\sqrt{1-\alpha_0^2}\rho) \\ &= \int_{-\infty}^{\infty}\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(\alpha_0)\frac{J_{n+\frac{1}{2}}(\lambda)}{\sqrt{\lambda}}\frac{J_{n+\frac{1}{2}}(\mu)}{\sqrt{\mu}}\right\}\exp(\imath\beta\lambda)d\lambda.\end{aligned} \tag{4.4}$$

The equation (4.4) is very useful because the sum within is evaluated by Hochstadt [16, page 221]),

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(\alpha_0)\frac{J_{n+\frac{1}{2}}(\lambda)}{\sqrt{\lambda}}\frac{J_{n+\frac{1}{2}}(\mu)}{\sqrt{\mu}} = \frac{1}{\pi}j_0(R), \tag{4.5}$$

where R is given by Hochstadt [16, page 219]

$$R = \sqrt{\lambda^2 + \mu^2 - 2\lambda\mu\alpha_0}$$
$$= \sqrt{\lambda^2 + \omega^2(z^2+\rho^2) - 2\lambda\omega\sqrt{z^2+\rho^2}\cos(\theta_0)}. \tag{4.6}$$

I have therefore reduced equation (4.4) to the integral representation of Bessel beams written below for any time t.

$$\exp(\imath\omega\alpha_0 z - \imath\omega t)J_0(\omega\sqrt{1-\alpha_0^2}\rho)$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} j_0(R)\exp[\frac{\imath z\lambda}{\sqrt{z^2+\rho^2}}]\exp(-\imath\omega t)d\lambda. \tag{4.7}$$

5. Expansion of Bessel Beam Wavepacket with Constant Spectrum

Bessel beam wavepacket with constant spectrum can be expanded into an infinite series of product of three Legendre polynomials. For this purpose let us integrate the partial wave expansion from (2.5) over variable ω in whole frequency domain:

(5.1)
$$\int_{-\infty}^{\infty} \exp[\imath\omega\cos(\theta)z - \imath\omega t]J_0[\omega\sin(\theta)\rho]d\omega$$
$$= \sum_{n=0}^{\infty} 2\imath^n(n+\frac{1}{2})P_n[\cos(\theta)]P_n[\tfrac{z}{\sqrt{z^2+\rho^2}}]\int_{-\infty}^{\infty} j_n(\omega\sqrt{z^2+\rho^2})\exp(-\imath\omega t)d\omega.$$

I also make the following variable substitutions

$$\cos(\eta) = \tfrac{z}{\sqrt{z^2+\rho^2}} \tag{5.2}$$

$$\omega = \frac{\mu}{\sqrt{z^2+\rho^2}}. \tag{5.3}$$

The right hand side of (5.1) then becomes

(5.4)
$$\frac{1}{\sqrt{z^2+\rho^2}}\sum_{n=0}^{\infty} 2\imath^n(n+\frac{1}{2})P_n[\cos(\theta)]P_n[\cos(\eta)]\int_{-\infty}^{\infty} j_n(\mu)\exp[\frac{-\imath\mu t}{\sqrt{z^2+\rho^2}}]d\mu.$$

I recognize the integral inside (5.4) as Fourier transform of Legendre polynomial namely

$$P_n[\frac{t}{\sqrt{z^2+\rho^2}}] = \frac{1}{\pi(-\imath)^n}\int_{-\infty}^{\infty} j_n(\mu)\exp[\frac{-\imath\mu t}{\sqrt{z^2+\rho^2}}]d\mu. \tag{5.5}$$

I let $cos(\gamma) = \frac{t}{\sqrt{z^2+\rho^2}}$ and therefore (5.1) becomes the desired expansion

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp[\imath\omega \cos(\theta) z - \imath\omega t] J_0[\omega \sin(\theta)\rho] d\omega \\ &= \frac{1}{\sqrt{z^2+\rho^2}} \sum_{n=0}^{\infty} 2\pi(n+\frac{1}{2}) P_n[\cos(\theta)] P_n[\cos(\eta)] P_n[\cos(\gamma)]. \end{aligned} \tag{5.6}$$

The left hand side of (5.6) is a superposition of all Bessel beams with frequencies ω from $-\infty$ to ∞, and whose frequency spectrum is $f(\omega) = 1$. The integral on the left hand side of (5.6) can be evaluated analytically as well and the result is:

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp[\imath\omega \cos(\theta) z - \imath\omega t] J_0[\omega \sin(\theta)\rho] d\omega \\ &= \frac{2}{\sqrt{\sin^2(\theta)\rho^2 - [t - \cos(\theta) z]^2}}. \end{aligned} \tag{5.7}$$

The integral result in equation (5.7) is valid if $| \ t - \cos(\theta) z \ | < \sin(\theta)\rho$, otherwise the integral is equal to zero. This means that the expansion in (5.6) vanishes if $\sin(\eta)\sin(\theta) < | \cos(\eta)\cos(\theta) - \cos(\gamma) \ |$. I therefore have evaluated the sum of the triple Legendre polynomial product as well:

$$\begin{aligned} &\sum_{n=0}^{\infty} 2\pi(n+\frac{1}{2}) P_n[\cos(\theta)] P_n[\cos(\eta)] P_n[\cos(\gamma)] \\ &= \frac{2\sqrt{z^2+\rho^2}}{\sqrt{\sin^2(\theta)\rho^2 - [t - \cos(\theta) z]^2}}. \end{aligned} \tag{5.8}$$

If I use the definitions for $\cos(\gamma)$ and $\cos(\eta)$, I finally get

$$\begin{aligned} &\sum_{n=0}^{\infty} 2\pi(n+\frac{1}{2}) P_n[\cos(\theta)] P_n[\cos(\eta)] P_n[\cos(\gamma)] \\ &= \frac{2}{\sqrt{\sin^2(\eta)\sin^2(\theta) - [\cos(\eta)\cos(\theta) - \cos(\gamma)]^2}}. \end{aligned} \tag{5.9}$$

The result of (5.9) is valid if $\sin(\eta)\sin(\theta) > | \cos(\eta)\cos(\theta) - \cos(\gamma) \ |$, otherwise the sum is equal to zero. The expansion of Bessel beam wavepacket with constant spectrum is a new result as well.

6. Bessel Beam Identity

In order to show that the two representations in (2.5) and (2.6) of Bessel beam are equivalent, I shall consider the following case: I will evaluate both representations along points of the cone only at $t = 0$ and I will integrate the results over all possible cones. Let me start with expansion first. At a

point on the cone I have $\cos(\eta) = \cos(\theta)$. I let $r = \sqrt{z^2 + \rho^2}$ and I write:

$$\begin{aligned}
&\int_{-1}^{1} \sum_{n=0}^{\infty} J_0[\omega r \sin^2(\theta)] \exp[\imath \omega r \cos^2(\theta)] d[\cos(\theta)] \\
&= \int_{-1}^{1} \sum_{n=0}^{\infty} 2\imath^n (n + \frac{1}{2}) j_n(\omega r) P_n[\cos(\theta)] P_n[\cos(\theta)] d[\cos(\theta)]. \qquad (6.1)
\end{aligned}$$

If I exploit the orthogonality of Legendre polynomials I get

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2\imath^n (n + \frac{1}{2}) j_n(\omega r) \int_{-1}^{1} P_n[\cos(\theta)] P_n[\cos(\theta)] d[\cos(\theta)] \\
&= \sum_{n=0}^{\infty} 2\imath^n (n + \frac{1}{2}) j_n(\omega r) \frac{1}{(n + \frac{1}{2})} \\
&= \sum_{n=0}^{\infty} 2\imath^n j_n(\omega r). \qquad (6.2)
\end{aligned}$$

Similarly, let me proceed wih integral representation:

$$\begin{aligned}
&\int_{-1}^{1} J_0[\omega r \sin^2(\theta)] \exp[\imath \omega r \cos^2(\theta)] d[\cos(\theta)] \\
&= \int_{-1}^{1} \frac{1}{\pi} \int_{-\infty}^{\infty} j_0(R) \exp[\frac{\imath z \lambda}{\sqrt{z^2 + \rho^2}}] d\lambda d[\cos(\theta)] \\
&= \int_{-1}^{1} \frac{1}{\pi} \int_{-\infty}^{\infty} j_0(R) \exp[\imath \lambda \cos(\gamma)] d\lambda d[\cos(\theta)]. \qquad (6.3)
\end{aligned}$$

The well known plane wave partial wave expansion namely [17]

$$\exp[\imath \lambda \cos(\gamma)] = \sum_{n=0}^{\infty} (n + \frac{1}{2}) \imath^n j_n(\lambda) P_n[\cos(\gamma)] \qquad (6.4)$$

is used to rewrite the last line of (6.3) as:

$$\begin{aligned}
&\int_{-1}^{1} \frac{1}{\pi} \int_{-\infty}^{\infty} j_0(R) \left\{ \sum_{n=0}^{\infty} (n + \frac{1}{2}) \imath^n j_n(\lambda) P_n[\cos(\gamma)] \right\} d\lambda d[\cos(\theta)] \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} (n + \frac{1}{2}) \imath^n \int_{-\infty}^{\infty} j_n(\lambda) \left\{ \int_{-\infty}^{\infty} j_0(R) P_n[\cos(\gamma)] d[\cos(\theta)] \right\} d\lambda. \qquad (6.5)
\end{aligned}$$

The inner integral of the right hand side of equation (6.5) can be evaluated [18] and therefore (6.5) reduces to:

$$\frac{1}{\pi} \sum_{n=0}^{\infty} (n + \frac{1}{2}) \imath^n \int_{-\infty}^{\infty} j_n(\lambda) \left\{ \frac{\pi}{\sqrt{\lambda \omega r}} J_{n+\frac{1}{2}}(\lambda) J_{n+\frac{1}{2}}(\omega r) \right\} d\lambda. \qquad (6.6)$$

Further simplification leads to

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})\imath^n j_n(\omega r)\int_{-\infty}^{\infty}\frac{[J_{n+\frac{1}{2}}(\lambda)]^2}{\lambda}d\lambda = \sum_{n=0}^{\infty}(n+\frac{1}{2})\imath^n j_n(\omega r)\frac{2}{\pi}\int_{-\infty}^{\infty}j_n(\lambda)^2 d\lambda. \tag{6.7}$$

The last integral in (6.7) can be evaluated as well:

$$\int_{-\infty}^{\infty}j_n(\lambda)^2 d\lambda = \frac{\pi}{2n+1}. \tag{6.8}$$

My final result therefore is

$$\int_{-1}^{1}J_0[\omega r\sin^2(\theta)]\exp[\imath\omega r\cos^2(\theta)]d[\cos(\theta)] = \sum_{n=0}^{\infty}2\imath^n j_n(\omega r) \tag{6.9}$$

which is the same result as in (6.2). The equivalence between the partial wave expansion and integral representation of Bessel beam is therefore verified. The result in (6.9) constitutes a new Bessel beam identity.

7. Partial Wave Expansion and Integral Representation with Dispersion

The importance of the partial wave expansion of Bessel beam becomes more obvious when dispersion is allowed into the wave equation. Dispersion arises when the index of refraction depends upon frequency. In this case the Bessel beam partial wave expansion becomes

(7.1)

$$\exp[\imath n(\omega)\omega\cos(\theta)z-\imath\omega t]J_0[n(\omega)\omega\sin(\theta)\rho] = \sum_{n=0}^{\infty}2\imath^n(n+\frac{1}{2})P_n[\cos(\theta)]P_n[\frac{z}{\sqrt{z^2+\rho^2}}]j_n[n(\omega)\omega\sqrt{z^2+\rho^2}]\exp[-\imath\omega t],$$

where $n(\omega)$ is frequency dependent index of refraction.

The index of refraction in the left hand side of (7.1) appears inside the Bessel function as well as inside the exponential, whereas $n(\omega)$ appears only inside the spherical Bessel function in the right hand side of (7.1). If a certain form of $n(\omega)$ allows the summation in (7.1) to be terminated at a certain integer value, then this may allow for better approximations. In the case of integral representation $n(\omega)$ is present inside the variable R only:

$$R = \sqrt{\lambda^2+n^2(\omega)\omega^2(z^2+\rho^2)-2\lambda n(\omega)\omega\sqrt{z^2+\rho^2}\cos(\theta)}. \tag{7.2}$$

8. Conclusion

In this paper, I gave a new proof for partial wave expansion of a Bessel beam, from a physical perspective. The partial wave expansion has been shown by Stratton [10, page 413] in a more general form. To my knowledge the integral representation of Bessel beam however is not found in literature. Furthermore the equivalency between the Bessel beam wavepacket with constant spectrum and the triple Legendre polynomial series as shown in section 5 is a new result as well. The similar sum has been found by Baranov [19] by using different more complicated method and our result coincides. The final result of section 6 further shows the equivalence between the partial wave expansion and integral representation of Bessel beams, which further confirms the integral representation. I have further developed the mathematical structure of Bessel beams with the hope that the experimental results in Bessel beam optics can be described more accurately and the controversy of superluminal propagation resolved.

References

[1] Durnin J 1987 Exact solutions for nondiffracting beams. I The scalar theory *J. Opt. Soc. Am* **4** 651–4

[2] Lu J and Greenleaf J F 1992 Nondiffracting X waves: Exact solutions to free-space scalar wave equation and their infinite realizations *IEEE Trans. Ultrasonic Ferroelectric Frequency Control* **39** 19–31

[3] Salo J, Fagerholm J, Friberg A T and Salomaa M M 2000 Unified description of nondiffracting X and Y waves *Phys. Rev. E* **62** 4261–75

[4] Hernndez-Figueroa H A, Zamboni-Rached M and Recami E 2008 *Localized Waves* (New Jersey: Wiley, John and Sons)

[5] Zamboni-Rached M, Recami E and Besieris I M 2010 Cherenkov radiation versus X-shaped localized waves *J. Opt. Soc. Am* **27** 928–34

[6] Walker S C and Kuperman W A 2007 Cherenkov-Vavliov formulation of X-waves *Phys. Rev. Lett.* **99** 244802

[7] McGloin D and Dholakia K 2005 Bessel beams: diffraction in a new light *Contemporary Physics* **46** 15–28

[8] Marston P L 2007 Scattering of Bessel beams *J. Acoust. Soc. Am.* **121** 2

[9] Mitri F G 2010 Gegenbauer expansion to model the incident wave-field of a high-order Bessel vortex beam in spherical coordinates *Ultrasonics* **50** 541–3

[10] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill)

[11] Neves A A R, Padilha L A, Fontes A, Rodriguez , Cruz C H B, Barbosa L C and Cesar C L 2006 Analytical results for a Bessel function times Legendre polynomials class integrals for nondiffracting beams *J.Phys. A: Math* **39** L293-6

[12] Cregg P J and Svendlindh P 2007 Comment on: Analytical results for a Bessel function times Legendre polynomials class integrals *J.Phys. A: Mat* **40** 14029–31

[13] Koumandos S 2007 On a class of integrals involving a Bessel function times Gegenbauer polynomials *Int. J. Math. Math. Sci.* 73750

[14] Dodonov V V 2007 Comment on: Analytical results for a Bessel function times Legendre polynomials class integrals *J.Phys. A: Mat* **40** 14329–30

[15] Mehrem R 2011 The plane wave expansion, infinite integrals and identities involving spherical Bessel functions *Appl. Math. Comp* **217** 5360–5

[16] Hochstadt H 1971 *Special Functions of Mathematical Physics* (Pure and Applied Mathematics Series vol 23) (New York: Wiley, John and Sons)

[17] Marion J B and Heald M A 1980 *Classical Electromagnetic Radiation* (New York: Academic) p 317
[18] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic) p 824
[19] Baranov A S 2006 On series containing products of Legendre polynomials *Mat. Zametki* **80** 171–8

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